Constructions of principal patches of Dupin cyclides defined by constraints: four vertices on a given circle and two perpendicular tangents at a vertex

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Abstract

Dupin cyclides are algebraic surfaces introduced for the first time in 1822 by the French mathematician Pierre-Charles Dupin. A Dupin cyclide can be defined as the envelope of an one-parameter family of oriented spheres, in two different ways. R. Martin is the first author who thought to use these surfaces in CAD/CAM and geometric modeling. Some authors gave algorithms to convert a Dupin cyclide patch into a rational biquadratic Bézier surface. Then, the control points of these Bézier surfaces have some properties. From these constraints, i.e. the four vertices are on a circle and two tangent vectors at a vertex are perpendicular, there are some algorithms which permit to construct and convert such a Bézier surface into a Dupin cyclide patch. In this paper, we use the space of spheres to simplify the construction of the Dupin cyclide patch. We do not determine the control points of the underlying Bézier surface.

1 Introduction

Dupin cyclides are algebraic surfaces introduced for the first time in 1822 by the French mathematician Pierre-Charles Dupin (see [28]) and were introduced in CAD by R. Martin in 1982 (see [17]). They have a low algebraic degree: at most 4. These Dupin cyclides have a parametric equation and two equivalent implicit equations (see [29, 23]) and they have been studied by a lot of mathematicians (see [22, 23, 2]). Since a score of years, much of authors used them in Computer Aided Geometric Design. In [18, 16, 19, 24], Dupin cyclides are represented by Rational Biquadratic Bézier Surfaces (RBBS). Then, the conditions of the control points of a RBBS which can represent a Dupin cyclide patch have been given. From the previous conditions, some constructions of RBBS which can be converted into Dupin cyclide patches have been proposed.
(see [11, 14, 8, 26]). To obtain an unique such RBBS, two initial conditions are necessary and sufficient (see [7, 15]): the four vertices belong to a circle; the tangent vectors at one of the four vertices, which define two circular edges, must be perpendicular.

The space of spheres was introduced in various ways. For example, M. Berger (see [2]) works in the projective space of the quadratic forms on the affine Euclidean space, M. Paluszny (see [31]) works in a four dimensional projective space using the hypersphere of Moebius while (see [30]), T. Cecil (see [27]), R. Langevin and P. Walczak (see [13]) use a four dimensional quadric $\Lambda^4$ in the five dimensional Lorentz space which is endowed with a non-degenerate indefinite quadratic form of signature $(4,1)$.

Some definitions of Dupin cyclides exist (see [20, 23, 29, 14]). The most appropriate in our context is: a Dupin cyclide can be defined, in two different ways, as the envelope of an one-parameter family of oriented spheres. The contribution of a sphere to the Dupin cyclide is a circle in the usual Euclidean 3D affine space $\mathbb{E}_3$ called characteristic circle. At each point belonging to this circle, the Dupin cyclide and the aforementioned sphere are tangent. In the space of spheres, a Dupin cyclide is represented by two conics in two perpendicular 2-planes. Using the transition formulae between these oriented spheres of $\mathbb{E}_3$ and the points of $\Lambda^4$, we can draw characteristic circles or arcs of characteristic circles of a given Dupin cyclide. Moreover, we can obtain a mesh of a Dupin cyclide patch using the conics in $\Lambda^4$ and we need not know its parameters nor the affine transformations to place the Dupin cyclide in the correct location of the scene (see [4, 9]).

In this paper, we compute a Dupin cyclide principal patch using the space of spheres. The initial conditions are given in [7, 15]: the four vertices belong to a circle and the tangents (in the 3D space) to two of the four edges, at one of the four vertices, are perpendicular. In comparison with [11, 8, 26], we do not need to: construct the RBBS, determine the type of the Dupin cyclide, the Dupin cyclide parameters, the transformation matrix. In fact, the visualization of the principal patch can be done directly from the space of spheres.

A problem exists with the iterative method which permits the constructions of Dupin cyclide principal patches: after four or five iterations, the control of the Dupin cyclide patches is not satisfactory (see [3]). Thus, unlike previous works (see [7, 15, 3, 11, 8, 26]), the use of the space of spheres to represent Dupin cyclide patches will permit to construct different meshes: we will be able to mix quadrilateral meshes and 3D triangular meshes with circular edges (see [11]). Moreover, we hope to use 3D triangular meshes without three circular edges: using the space of spheres, from three contact conditions, we can determine a one-parameter family of Dupin cyclides tangent along a common curve (see [6, 5]).

The paper is organised as follow: after some background about Dupin cy-
clides, the space of spheres and pencils of spheres, we give, in section 3, a theorem and an algorithm to compute a Dupin cyclide principal patch. After the conclusion and the perspectives, Appendix 5 contains the proof of the aforementioned theorem.

2 Background

2.1 Non-degenerate Dupin cyclides in the usual Euclidean $3D$ spaces $\mathbb{E}_3$

Let $(O_3, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be the orthonormal basis frame of the usual Euclidean space $\mathbb{E}_3$. In the Euclidean affine space $\mathbb{E}_3$, each sphere $S$, with a non-negative radius, defines two oriented spheres $S^+$ and $S^-$. In fact, we distinguish the inside space and the outside space of $S$. To define an oriented sphere with centre $C$ and radius $\rho$, we use an algebraic radius which is positive (respectively negative) if the unit normal vector $\mathbf{N}$ at the point $M$ is in the same direction as (respectively opposite direction to) than the vector $\mathbf{CM}$ and we have

$$\mathbf{CM} = \rho \mathbf{N}$$

Using Table 1, a parametric equation of a Dupin cyclide is

$$\Gamma_d(\theta, \psi) = \left( \begin{array}{c} \mu (c - a \cos \theta \cos \psi) + b^2 \cos \theta \\ a - c \cos \theta \cos \psi \\ b \sin \theta \times (a - \mu \cos \psi) \\ a - c \cos \theta \cos \psi \\ b \sin \psi \times (c \cos \theta - \mu) \\ a - c \cos \theta \cos \psi \end{array} \right)$$

where the pair $(\theta, \psi)$ belongs to $[0, 2\pi]^2$.

We can note that a Dupin cyclide is characterized by a third independent parameter $\mu$ (see [18, 12]). We can distinguish five kinds of Dupin cyclides: ring, Figure 3(a); spindle, Figure 3(e); horned, Figure 3(d); one-singularity
Figure 1. A Dupin cyclide, envelope of two families of one parameter spheres where the locus of the centres are an ellipse and a hyperbola in two orthogonal planes: (a) the centres of the spheres belong to an ellipse, (b) the centres of the spheres belong to a hyperbola (two planes belong to the family)

<table>
<thead>
<tr>
<th>Object</th>
<th>About the ellipse $E$</th>
<th>About the hyperbola $H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spheres</td>
<td>$S_1(\theta)$</td>
<td>$S_2(\psi)$</td>
</tr>
<tr>
<td>Centres</td>
<td>$\Omega_{\theta}(a \cos(\theta), b \sin(\theta), 0)$ \hspace{1cm} (2.2)</td>
<td>$\Omega_{\psi}\left(\frac{c}{\cos(\psi)}, 0, -b \tan(\psi)\right)$ \hspace{1cm} (2.3)</td>
</tr>
<tr>
<td>Radii</td>
<td>$r_1(\theta) = \mu - c \cos(\theta)$ \hspace{1cm} (2.4)</td>
<td>$r_2(\psi) = \mu - \frac{a}{\cos(\psi)}$ \hspace{1cm} (2.5)</td>
</tr>
<tr>
<td>Planes</td>
<td>$cx - \mu a = \varepsilon b z$</td>
<td></td>
</tr>
</tbody>
</table>

Table 1. Properties of the Dupin cyclides: spheres and planes belonging to each family, $\theta \in [0, 2\pi]$, $\psi \in [0, 2\pi] \setminus \left\{ -\frac{\pi}{2}, \frac{\pi}{2} \right\}$ and $\varepsilon \in \{-1, 1\}$
Figure 2. Orientations of a ring Dupin cyclide (CD4A) and a horned Dupin cyclide (CD4E) which are tangent along two principal circles $C_1$ and $C_2$ in the plane $\mathcal{P}_y$ of equation $y = 0$. The centre of $C_1$ (respectively $C_2$) is $O_1$ (respectively $O_2$) whereas the radius of $C_1$ (respectively $C_2$) is $r_1$ (respectively $r_2$).

$(a)$ $(b)$ $(c)$

Figure 3. Five types of Dupin cyclides in $\mathcal{E}_3$: (a) ring, (b) horned, (c) singly horned, (d) spindle, (e) one-singularity spindle.
spindle, Figure 3(e) and singly horned, Figure 3(d). Note that one can find some other definitions of Dupin cyclides (see [23, 29]).

Figure 2 shows two principal circles (Principal circles are characteristic circles belonging to the planes of equations $y = 0$ or $z = 0$ and two coplanar principal circles permit the computation of the Dupin cyclide parameters.) $\mathcal{C}_1$ and $\mathcal{C}_2$ in the plane of equation $y = 0$ for a ring Dupin cyclide, called CD4A, and a horned Dupin cyclide, called CD4E. The centre of $\mathcal{C}_1$ (respectively $\mathcal{C}_2$) is $O_1$ (respectively $O_2$) whereas the radius of $\mathcal{C}_1$ (respectively $\mathcal{C}_2$) is $r_1$ (respectively $r_2$). The same formula permits the computation of the parameter $a$ by

$$a = \frac{||\overrightarrow{O_1O_2}||}{2}$$

whereas the parameters $\mu$ and $c$ are solutions of the following system

$$\begin{cases} 
\mu + c &= r_1 \\
|\mu - c| &= r_2 
\end{cases}$$

(2.7)

We can note that

- along the circle $\mathcal{C}_1$, the tangent spheres which give the characteristic circles of each Dupin cyclides have the same orientation. This is schematized by the vector $\overrightarrow{n_1}$;

- along the circle $\mathcal{C}_2$, the orientations of the spheres which give a characteristic circle of each Dupin cyclides, are opposite. This is schematized by the vectors $\overrightarrow{n_2^A}$ for the CD4A and $\overrightarrow{n_2^E}$ for the CD4E, see Formula (2.1).

Using the notion of oriented spheres, we have $\rho_1 = r_1$ and $\rho_2 = -r_2$. Thus, the system of the Formula (2.7) becomes

$$\begin{cases} 
\mu + c &= \rho_1 \\
\mu - c &= \rho_2 
\end{cases}$$

(2.8)

and we do not need absolute values.

We note that, in the horned Dupin cyclide case, the change of orientation of the spheres leads to two singular points (i.e. two spheres having a radius equal to zero). To obtain the relation between the types of the Dupin cyclides and the parameter values, one can see [14].
Definition | Nature in $L_{4,1}$ | Euclidean view
--- | --- | ---
$C_l = \left\{ M \in L_{4,1} \mid \mathcal{P}_{4,1} \left( \overrightarrow{O_5 M} \right) = 0 \right\}$ | Sphere with a null radius | Light cone

(2.11)

$\Lambda^4 = \left\{ M \in L_{4,1} \mid \mathcal{P}_{4,1} \left( \overrightarrow{O_5 M} \right) = 1 \right\}$ | Unit sphere of one sheet | Hyperboloid

(2.12)

| Table 2. Fundamental quadrics of $L_{4,1}$ |

2.2 Construction of the space $\Lambda^4$ of oriented spheres and planes of $E_3$

2.2.1 The Lorentz space

Let $\overrightarrow{L_{4,1}}$ be the 5 dimensional vector space of vector basis $(\overrightarrow{e_i})_{i \in [0,4]}$, equipped with the non-degenerate indefinite symmetric bilinear form $\mathcal{L}_{4,1}$ of signature $(4,1)$ defined by

$$\mathcal{L}_{4,1} : \overrightarrow{L_{4,1}} \times \overrightarrow{L_{4,1}} \rightarrow \mathbb{R}$$

$$\left( \overrightarrow{u}, \overrightarrow{v} \right) \rightarrow -x_0y_0 + \sum_{i=1}^{4} x_iy_i$$

(2.9)

where $\overrightarrow{u} \left( x_0, \ldots, x_4 \right)$ and $\overrightarrow{v} \left( y_0, \ldots, y_4 \right)$ lie in $\overrightarrow{L_{4,1}}$. Let $\mathcal{P}_{4,1}$ be the non-degenerate indefinite quadratic form associated to $\mathcal{L}_{4,1}$ i.e.

$$\mathcal{P}_{4,1} \left( \overrightarrow{u} \right) = \mathcal{L}_{4,1} \left( \overrightarrow{u}, \overrightarrow{u} \right)$$

(2.10)

Let $L_{4,1}$ be the affine space associated to the vector space $\overrightarrow{L_{4,1}}$ with the origin $O_5 \left( 0, 0, 0, 0, 0 \right)$. In the space $L_{4,1}$, two quadrics play an important role: $\Lambda^4$ and $C_l$, Table 2. The set $C_l$, called light cone, is the quadric of the isotropic position vectors (for the Lorentz form) while the set $\Lambda^4$ represents the space of oriented spheres and planes of $E_3$. To explain the construction of this space, we have to define the different types of vectors and planes, Tables 3 and 4.

2.2.2 Embedding of $E_3$ on the light cone

We have to build a model of the ambient Euclidean affine space $E_3$ embedded as a sub-manifold of $L_{4,1}$ in order to manipulate simultaneously the spheres.
Constructions of Dupin cyclides

<table>
<thead>
<tr>
<th>Vector types $\overrightarrow{v}$</th>
<th>Space</th>
<th>Time</th>
<th>Light</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{D}_{4,1}(\overrightarrow{v}) &gt; 0$</td>
<td>$\mathcal{D}_{4,1}(\overrightarrow{v}) &lt; 0$</td>
<td>$\mathcal{D}_{4,1}(\overrightarrow{v}) = 0$</td>
<td></td>
</tr>
</tbody>
</table>

**Table 3.** The three different types of vectors $\overrightarrow{v}$ of the vector space $\overrightarrow{L}_{4,1}$

<table>
<thead>
<tr>
<th>Type</th>
<th>2-dimensional plane called 2-plane</th>
</tr>
</thead>
<tbody>
<tr>
<td>Space</td>
<td>All the vectors are space-like vectors</td>
</tr>
<tr>
<td>Time</td>
<td>Contain at least a time-like vector or exactly two non-collinear light ones</td>
</tr>
<tr>
<td>Light</td>
<td>Planes parallel to a tangent plane to $C_l$</td>
</tr>
</tbody>
</table>

**Table 4.** Three types of 2-planes of $\overrightarrow{L}_{4,1}$ and $L_{4,1}$

and planes of $\mathcal{E}_3$ and the points of $\mathcal{E}_3$. To do that, we have to build an isometry between $\mathcal{E}_3$ and a paraboloid $P$, section of the cone $C_l$ by a particular affine hyperplane of $L_{4,1}$ and we obtain a one-to-one relation between the oriented spheres of $\mathcal{E}_3$ and the points of $\Lambda^4$.

For example, choose two vectors $\overrightarrow{m_1}(1,0,0,0,1)$ and $\overrightarrow{m_2}(\frac{1}{2},0,0,0,-\frac{1}{2})$ and the hyperplane $H$, parallel to $\overrightarrow{m_1}$ (i.e. the space of vectors $\mathcal{L}_{4,1}$-orthogonal – in the rest of this paper, the term $\mathcal{L}_{4,1}$ is implied) to $\overrightarrow{m_1}$) passing through the point $m_2(1,0,0,1,0)$, Figure 4. The hyperplane $H$ is defined by the point $m_2$, the three space-like vectors $\overrightarrow{e_1}$, $\overrightarrow{e_2}$ and $\overrightarrow{e_3}$ and the light-like vector $\overrightarrow{m_1} = \overrightarrow{e_0} + \overrightarrow{e_4}$ and then, the equation of $H$ is $x_0 - x_4 = 1$. The equation of the paraboloid $P$ is

$$\begin{cases} 
  x_0 - x_4 - 1 = 0 \\
  x_0 - \frac{1}{2} (x_1^2 + x_2^2 + x_3^2 + 1) = 0 
\end{cases}$$

The formulae of the one-to-one correspondence $\Pi$ between $\mathcal{E}_3$ and $P$ are given in Table 5, (see [12]). So, we identify the point $M(x, y, z)$ of $\mathcal{E}_3$ with the point $\Pi(M)$ of $P$. Then, the origin $O_3$ of $\mathcal{E}_3$ in $L_{4,1}$ is defined by $\overrightarrow{O_5 O_3} = \overrightarrow{m_2}$.

Moreover, each light vector $\overrightarrow{m}$, which is not parallel to $\overrightarrow{m_1}$, defines a point $m$ on the paraboloid $P$ (m is the intersection between $P$ and the line, passing through $O_5$, generated by $\overrightarrow{m}$) and from this point $m$, we obtain a point $M$ in $\mathcal{E}_3$.

### 2.2.3 The space of spheres $\Lambda^4$

Let $n$ be in the set $[1,4]$. A $n$-submanifold $\mathcal{V}$ in $L_{4,1}$ is a $n$-surface without singular point. One can generalize Table 4 to $\mathcal{V}$ replacing the term 2-plane by tangent space, called $T_\sigma \mathcal{V}$, to $\mathcal{V}$ at $\sigma$. If, for all $\sigma$ in $\mathcal{V}$, the type of $T_\sigma \mathcal{V}$ is the same, one can distinguish submanifolds of $L_{4,1}$. As it had been understood by G. Darboux, the space of the oriented spheres (the planes are particular spheres) is represented by the quadric $\Lambda^4$, subset of $L_{4,1}$, Formula (2.12). This quadric...
Figure 4. Construction of the 3-dimensional paraboloid $P$ isometric to $E_3$, the hyperplane $\mathcal{H}$ is tangent to $C_1$: Each light-like vector $\vec{m}$, non-parallel to $\vec{n}_1$, corresponds to a point of $E_3$ via the paraboloid $P$: this point is the intersection between the line $(O, \vec{m})$ and $P$.

<table>
<thead>
<tr>
<th>Point of $E_3$</th>
<th>Direction</th>
<th>Point of $P \subset L_{4,1}$</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M = \begin{pmatrix} x \ y \ z \end{pmatrix}$</td>
<td>$\rightarrow$</td>
<td>$m = \begin{pmatrix} \frac{1}{2}(x^2 + y^2 + z^2 + 1) \ x \ y \ z \ \frac{1}{2}(x^2 + y^2 + z^2 - 1) \end{pmatrix}$</td>
<td>$x_0 = x_4 + 1$</td>
</tr>
<tr>
<td>$M = \begin{pmatrix} x_1 \ x_2 \ x_3 \end{pmatrix}$</td>
<td>$\leftarrow$</td>
<td>$m = \begin{pmatrix} x_0 \ x_1 \ x_2 \ x_3 \ x_4 \end{pmatrix}$</td>
<td>$x_0 = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2 + 1)$</td>
</tr>
</tbody>
</table>

Table 5. Correspondence between the points of $E_3$ and the points of the paraboloid $P$. 
is an unitary sphere with centre $O_5$ (for the Lorentz form) and a submanifold of type $(3, 1)$. Indeed, at each point $\sigma$ of $\Lambda^4$, the tangent hyperplane $T_\sigma \Lambda^4$ to $\Lambda^4$ is the set of points $M$ verifying

$$\mathcal{L}_{4,1} \left( \overrightarrow{O_5 \sigma}, \sigma M \right) = 0$$

(2.14)

We can note this hyperplane is time-like: the signature of the restriction of $\mathcal{Q}_{4,1}$ to this hyperplane is $(3, 1)$. Formula (2.14), which is equivalent to

$$\mathcal{L}_{4,1} \left( \overrightarrow{O_5 \sigma}, \overrightarrow{O_5 M} \right) = 1$$

(2.15)

shows that the vector $\overrightarrow{O_5 \sigma}$ and the gradient to $T_\sigma \Lambda^4$ at a point $\sigma$ are colinear (see Formula (2.1), for the particular case of $\mathcal{E}_3$). The deduction is easy using the Chasles relation and the formula

$$\mathcal{L}_{4,1} \left( \overrightarrow{O_5 \sigma}, \overrightarrow{O_5} \sigma \right) = -1$$

In the same way, a curve on $\Lambda^4$ can have a type. Our interest is the space-like curve i.e. at each point, the tangent vector is space-like. One can note too that $\Lambda^4$ contains a lot of affine lines which are light-like. Indeed, for each point $\sigma$ on $\Lambda^4$, the intersection between $\Lambda^4$ and $T_\sigma \Lambda^4$ is a light cone of dimension 3 (all the generatrices are light-like).

The relation between the points on $\Lambda^4$ and the oriented spheres in $\mathcal{E}_3$ is obtained using the $\mathcal{L}_{4,1}$-orthogonality: if $\sigma$ belongs to $\Lambda^4$, then the intersection between the hyperplane parallel to $T_\sigma \Lambda^4$ passing through $O_5$ and the paraboloid $P$ is a sphere or a plane. These calculations are left to the readers.

It is possible to prove (see [13]) that two spheres $S_1$ and $S_2$, with the same orientation at their contact point $M$, are tangent if and only if we have

$$\mathcal{Q}_{4,1} (\sigma_1, \sigma_2) = 0$$

where $\sigma_1$ and $\sigma_2$ are the representations, in the space of spheres, of the spheres $S_1$ and $S_2$ respectively. Moreover, the point $M$ is defined by the intersection between the paraboloid $P$ and the light-like line generated by the point $O_5$ and the vector $\sigma_1\sigma_2$.

Let $\mathcal{H}_{x_0=x_4}$ be the hyperplane of equation $x_0 = x_4$. This hyperplane permits to distinguish the spheres and the planes. Let $\Omega(a, b, c)$ (respectively $r$) be the centre (respectively the algebraic radius) of an oriented sphere in $\mathcal{E}_3$. The representation of this sphere on $\Lambda^4$ is

$$\sigma = \frac{1}{r} \left( \frac{a^2 + b^2 + c^2 - r^2 + 1}{2}, a, b, c, \frac{a^2 + b^2 + c^2 - r^2 - 1}{2} \right)$$

(2.16)
and, it is very easy to remark the two different oriented spheres, defined from the same geometrical sphere, are symmetric with respect to $O_5$ on the one hand, and $\sigma$ does not belong to the hyperplane $H_{x_0=x_4}$ on the other hand. The representation of an oriented 2-plane of equation

$$ax + by + cz = d, \quad \text{with} \quad \sqrt{a^2 + b^2 + c^2} = 1 \quad (2.17)$$

is

$$\sigma = (d,a,b,c,d) \quad (2.18)$$

and

- the point $\sigma$ belongs to the hyperplane $H_{x_0=x_4}$;
- the condition given in Formula (2.17) ensures the point $\sigma$ belongs to $\Lambda^4$ i.e. we have

$$\mathcal{Q}_{4,1}(O_5\sigma) = 1$$

Reciprocally, each point $\sigma(x_0,x_1,x_2,x_3,x_4)$ on $\Lambda^4$ defines an oriented plane or an oriented sphere. If we have $x_0 = x_4$, $\sigma$ represents the affine plane (in $E_3$) of equation

$$x_1x + x_2y + x_3z = x_0 \quad (2.19)$$

whereas if we have $x_0 \neq x_4$, $\sigma$ represents the oriented sphere (in $E_3$), its centre is

$$\Omega\left(\frac{x_1}{x_0-x_4}, \frac{x_2}{x_0-x_4}, \frac{x_3}{x_0-x_4}\right) \quad (2.20)$$

and its algebraic radius is

$$\frac{1}{x_0-x_4} \quad (2.21)$$

### 2.2.4 Representation of Dupin cyclides in the space of spheres

First, let us recall a Dupin cyclide is the envelope of two families of one-parameter oriented spheres. The representation of a Dupin cyclide in the space of spheres $\Lambda^4$ is the union of two circles or the union of a circle and a line for the quadratic form $\mathcal{Q}_{4,1}$ (see [12]).

Using an Euclidean construction, we can see these conics as:

- two ellipses, figure 5(a), for a ring Dupin cyclide;
- an ellipse and a hyperbola, figure 5(b), for a spindle or horned Dupin cyclide;
- an ellipse and a parabola, figure 5(c), for a singly horned Dupin cyclide or one-singularity spindle Dupin cyclide.
Constructions of Dupin cyclides

Figure 5. Representation of Dupin cyclides on $\Lambda^4$: (a) a ring Dupin cyclide is two connected circles, (b) a Dupin cyclide with two singular points is the union of two circles, one is connected, the other has two connected components, (c) a Dupin cyclide with one only singular point is the union of a circle and a line

We note in the space of spheres, the tangency directions to these conics are space-like directions in the hyperplane tangent to the quadric $\Lambda^4$. Obviously, if the kind of tangency directions, generated by a vector $\overrightarrow{v_1}$, was time-like, the intersection between the half-line $[O_5, \overrightarrow{v_1})$ and $\Lambda^4$ would be the empty set. The characteristic circles of a Dupin cyclide can be determined by the computation of the intersection of two orthogonal spheres (see [4, 9]). Let $t \mapsto \sigma_1(t)$ be the parametrisation of the conic representing one of the two families of spheres. The representation of the first sphere $S(t_0)$ in $E_3$ is the point $\sigma_0 = \sigma_1(t_0)$ in $\Lambda^4$. The second sphere $\sigma_1(t_0)$ is given using the tangent vector $\frac{d\sigma}{dt}(t_0)$ to the curve at the point $\sigma(t_0)$ and we have

$$\left\{ \sigma(t_0) \right\} = \left[ O_5, \frac{d\sigma}{dt}(t_0) \right] \cap \Lambda^4$$

(2.22)

2.2.5 Linear pencils of spheres and their correspondence in $\Lambda^4$

In [13], the authors recall that any linear pencil of spheres of $E_3$ is represented in $L_{4,1}$ by the section of the quadric $\Lambda^4$ by an affine 2-plane $P$ passing through $O_5$. Given the type of the plane $P$, we have different types of pencil of spheres

Proposition 1:

- The section of the quadric $\Lambda^4$ by a space-like plane $P$ corresponds to a pencil of spheres with a base circle i.e. all the spheres of the pencil have a common circle, Figures 6(a) and 7(a).
Figure 6. Three pencils of spheres in $E_3$: (a) a pencil of spheres with a base circle, (b) a Poncelet pencil, (c) A pencil of tangent spheres at a point.

- The section of the quadric $Λ^4$ by a time-like plane $P$ corresponds to a Poncelet pencil, i.e. the balls, defined by the spheres, are contained in each other and tend to two limit points, Figures 6(b) and 7(b).

- The section of the quadric $Λ^4$ by a light-like plane $P$ corresponds to a pencil of tangent spheres in a point is the union of two light-like lines, Figures 6(c) and 7(c).

3 Algorithm to construct a Dupin cyclide principal patch

3.1 The constraints in $E_3$

Let us recall the definition of Rational Biquadratic Bézier Surfaces (RBBS). A point $M(u,v), (u,v) \in [0,1]^2$, belongs to the RBBS defined by the weighted points $(P_{ij}, \omega_{ij})_{(i,j) \in [0,2]^2}$, if we have the relation

$$OM(u,v) = \frac{1}{\sum_{i=0}^{2} \sum_{j=0}^{2} w_{ij} B_i(u) B_j(v) OP_{ij}} \sum_{i=0}^{2} \sum_{j=0}^{2} \omega_{ij} B_i(u) B_j(v) OP_{ij}$$  (3.1)
Figure 7. Representation of three pencils of spheres in $\Lambda^4$, each pencil is the section of $\Lambda^4$ by a 2-plane: (a) a pencil of spheres with a base circle and the type of the 2-plane is space-like, (b) a Poncelet pencil and the type of the 2-plane is time-like, (c) a pencil of tangent spheres at a point and the type of the 2-plane is light-like.

<table>
<thead>
<tr>
<th>Name</th>
<th>Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>(PG1)</td>
<td>$P_{00}, P_{02}, P_{22}$ and $P_{20}$ are cocyclic (i.e. form a cyclic quadrilateral)</td>
</tr>
<tr>
<td>(PG2)</td>
<td>$</td>
</tr>
<tr>
<td>(PG3)</td>
<td>$P_{00}P_{10} \bullet P_{00}P_{01} = 0$, $P_{02}P_{10} \bullet P_{02}P_{12} = 0$, $P_{22}P_{12} \bullet P_{22}P_{21} = 0$, $P_{20}P_{21} \bullet P_{20}P_{10} = 0$</td>
</tr>
<tr>
<td>(PG4)</td>
<td>$P_{11} \in \text{Aff}{P_{00}, P_{01}, P_{10}} \cap \text{Aff}{P_{02}, P_{12}, P_{01}} \cap \text{Aff}{P_{20}, P_{21}, P_{10}} \cap \text{Aff}{P_{22}, P_{21}, P_{12}}$</td>
</tr>
</tbody>
</table>

Table 6. Geometrical properties of a rational biquadratic Bézier surface obtained by conversion of a Dupin cyclide, the notation $\mathbf{u} \bullet \mathbf{v}$ designates the Euclidean dot product between $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ and $\text{Aff}\{A, B, C\}$ designates the affine space generated by the points $A$, $B$ and $C$.

where

$$B_0(t) = (1 - t)^2, \quad B_1(t) = 2t(1 - t) \quad \text{and} \quad B_2(t) = t^2$$

are the quadratic Bernstein polynomials (see [18]).

Several authors gave algorithms to convert a Dupin cyclide patch into a rational biquadratic Bézier surface (see [18, 16, 25, 14]) and we can cite some of the properties of the control points, Table 6. These conditions are necessary and sufficient.

We can note that a Dupin cyclide principal patch is defined by six points $P_{00}, P_{20}, P_{22}, P_{02}, P_{10}$ and one of the two following points $P_{11}$ or $P_{01}$ (see [1, 8]) and the constraints:

- the points $P_{00}, P_{20}, P_{22}, P_{02}$ form a cyclic quadrilateral (see [17] and
the tangent lines \((P_{00}P_{10})\) and \((P_{00}P_{01})\) to the RBBS at \(P_{00}\) are perpendicular.

3.2 Transfer of the constraints in \(E^3\) onto \(\Lambda^4\)

From the previous section, we start with four points \(M_{00}, M_{01}, M_{11}\) and \(M_{10}\) on a circle and two perpendicular tangents vectors \(\vec{v}_{C_{\theta_0}}\) and \(\vec{v}_{C_{\psi_0}}\) at \(M_{00}\) (see [7, 15]), Figure 8. The vectors \(\vec{v}_{C_{\theta_0}}\) and \(\vec{v}_{C_{\psi_0}}\) replace the previous tangent lines. The points \(M_{10}, M_{01}\) (respectively \(M_{11}\)) and the vector \(\vec{v}_{C_{\theta_0}}\) (respectively \(\vec{v}_{C_{\psi_0}}\)) define the circle \(C_{\theta_0}\) (respectively \(C_{\psi_0}\)).

Each circle of \(E^3\) permits to construct a pencil of spheres with a base circle \(C\) (\(C_{\theta_0}\) or \(C_{\psi_0}\)) and two elements of this family are particular: one is an oriented sphere having \(C\) as great circle and the second is an oriented plane passing through \(C\). We can remark this sphere and this plane are perpendicular.

First, in the space of spheres, each pencil of spheres is represented by a circle obtained as the section of \(\Lambda^4\) by a space-like plane. The point \(O_5\) belongs to this plane and is the centre of the circle. Then, using Theorems 1 and 2, we compute a point \(\sigma_{\theta_0}\) on one circle and a point \(\sigma_{\psi_0}\) on the other circle such as their representations in \(E^3\) are to spheres tangent at \(M_{00}\).

Second, we build the light-like line defined by the point \(\sigma_{\psi_0}\) and the vector \(\vec{m}_{10}\) which is the representation of the point \(M_{10}\). Third, we compute the point \(\sigma_{\theta_2}\) on this light-like line such that its representation in \(E^3\) is a sphere containing the point \(M_{11}\).

In [15], the data are four points on a circle, a tangent vector to a characteristic circle \(C_{\theta_0}\) at \(M_{00}\) and a normal vector to the surface at \(M_{00}\). The authors use some iterations to build the other characteristic circles. The sphere \(S_{\theta_0}\) which contains the circle \(C_{\theta_0}\) is computed. The vector tangent which determines the second characteristic circle \(C_{\psi_0}\) is the cross product between the two previous vectors.

Then, the sphere \(S_{\psi_0}\), containing the circle \(C_{\psi_0}\) is computed. The two other circles \(C_{\theta_2}\) and \(C_{\psi_2}\), the two spheres \(S_{\theta_2}\) and \(S_{\psi_2}\) are built in the same way.

In the space of spheres, our method permits to determine the adequate spheres and we do not calculate intersections of straight lines. We directly compute the sphere \(S_{\theta_0}\) (respectively \(S_{\psi_0}\)) from the pencils of spheres with a base circle \(C_{\theta_0}\) (respectively \(C_{\psi_0}\)). Then, we determine the sphere \(S_{\theta_2}\) which generates the characteristic circle \(C_{\theta_2}\). To summarize, we just need three spheres to determine the Dupin cyclide.

Some preliminary results must be given before the presentation of Algorithm 1: Theorem 1 characterises the relative positions of two spheres using
FIGURE 8. Conditions, in $\mathbb{S}_3$, to obtain a Dupin cyclide principal patch, the points $M_{00}$, $M_{01}$, $M_{11}$ and $M_{10}$ belong to a circle (i.e. form a cyclic quadrilateral): The points $M_{00}$, $M_{01}$ (respectively $M_{10}$) and the vector $\overrightarrow{v_{C\theta_0}}$ (respectively $\overrightarrow{v_{C\psi_0}}$) define a circular arc $C_{\theta_0}$ (respectively $C_{\psi_0}$): The vectors $\overrightarrow{v_{C\theta_0}}$ and $\overrightarrow{v_{C\psi_0}}$ are perpendicular.
the Lorentz form and their representations in $\Lambda^4$, Theorem 2, proved in Appendix I, permits the computation of the spheres $S_{\theta_0}$ and $S_{\nu_0}$.

### 3.3 Relative positions of two spheres

**Theorem 1**: Relative positions of two spheres

Let $S$ and $S_x$ be two oriented spheres in $E_3$. Let $\sigma$ and $\sigma_x$ be their representations on $\Lambda^4$.

Then, we have three cases:

- $S \cap S_x$ is a circle if and only if $|L_{4,1}(\overrightarrow{O_5\sigma}, \overrightarrow{O_5\sigma_x})| < 1$

- $S$ and $S_x$ are tangent if and only if $|L_{4,1}(\overrightarrow{O_5\sigma}, \overrightarrow{O_5\sigma_x})| = 1$

- $S \cap S_x = \emptyset$ if and only if $|L_{4,1}(\overrightarrow{O_5\sigma}, \overrightarrow{O_5\sigma_x})| > 1$

**Proof**: Let $\mathcal{P}$ be the affine 2-plane generated by the points $O_5$, $\sigma$, and $\sigma_x$. Let $\overrightarrow{\mathcal{P}}$ be the vector plane defined by the vectors $\overrightarrow{O_5\sigma_x}$ and $\overrightarrow{O_5\sigma}$. Let $\overrightarrow{u}$ be a vector belonging to $\overrightarrow{\mathcal{P}}$ such as

- $\overrightarrow{u}$ and $\overrightarrow{O_5\sigma}$ are perpendicular i.e.

  $$L_{4,1}(\overrightarrow{u}, \overrightarrow{O_5\sigma_x}) = 0$$

- if $\overrightarrow{u}$ is not a light-like vector, we have $|Q_{4,1}(\overrightarrow{u})| = 1$.

Thus, we have the following property

$$\exists (\alpha, \beta) \in \mathbb{R}^2 \mid \overrightarrow{O_5\sigma} = \alpha \overrightarrow{O_5\sigma_x} + \beta \overrightarrow{u}$$

Since $\overrightarrow{O_5\sigma_x}$ is a space-like vector on the one hand, $\overrightarrow{u}$ and $\overrightarrow{O_5\sigma}$ are perpendicular on the other hand, the type of the vector $\overrightarrow{u}$ determine the type of the plane $\overrightarrow{\mathcal{P}}$

- $Q_{4,1}(\overrightarrow{u}) = 1$ implies that the type of $\overrightarrow{\mathcal{P}}$ is space-like and then the pencil of spheres is a pencil of spheres with a base circle,

- $Q_{4,1}(\overrightarrow{u}) = 0$ implies that the type of $\overrightarrow{\mathcal{P}}$ is light-like and then the pencil of spheres is a pencil of tangent spheres,

- $Q_{4,1}(\overrightarrow{u}) = -1$ implies that the type of $\overrightarrow{\mathcal{P}}$ is time-like and then the pencil of spheres is a Poncelet pencil.
Morover, as \( \sigma \) and \( \sigma_x \) belong to \( \Lambda^4 \), from formulae (3.4) and (3.3), we obtain

\[
1 = \mathcal{D}_{4,1} (\overrightarrow{O_5 \sigma}) = \alpha^2 + \beta^2 \mathcal{D}_{4,1} (\overrightarrow{u})
\]  (3.5)

The Formula (3.5) permits to compare \( |\alpha| \) to 1 according to the type of \( \overrightarrow{u} \), Table 7. Since

\[
\mathcal{L}_{4,1} (\overrightarrow{O_5 \sigma}, \overrightarrow{O_5 \sigma_x}) = \alpha
\]  (3.6)

we have the expected result. ■

We can note for a Dupin cyclide, if \( \sigma_E \) (respectively \( \sigma_H \)) represents an oriented sphere centered on the ellipse \( E \) (respectively hyperbola \( H \)), then the type of the line \( (\sigma_E, \sigma_H) \) is light-like and we have

\[
\mathcal{L}_{4,1} (\overrightarrow{O_5 \sigma_E}, \overrightarrow{O_5 \sigma_H}) = 1
\]  (3.7)

### 3.4 Theorem and algorithm

We can remark that if \( C \) is a great circle on the oriented sphere \( S \), and if \( \mathcal{P} \) is an oriented plane containing the circle \( C \), then, an expression (on \( \Lambda^4 \)) of a sphere which belongs to the pencil of spheres with \( C \) as base circle is

\[
\overrightarrow{O_5 \gamma_0} (\theta) = \cos (\theta) \overrightarrow{O_5 \sigma_{\theta_j}} + \sin (\theta) \overrightarrow{O_5 \sigma_{\theta_i}}
\]  (3.8)

where \( \theta \in [0, 2\pi] \) and \( \sigma_{\theta_j} \) (respectively \( \sigma_{\theta_i} \)) is the representation of the oriented sphere \( S \) (respectively the oriented plane \( \mathcal{P} \)). Moreover, we have

\[
\mathcal{L}_{4,1} (\overrightarrow{O_5 \sigma_{\theta_j}}, \overrightarrow{O_5 \sigma_{\theta_i}}) = 0
\]

Let us recall that the points \( M_{00} \) and \( M_{01} \) and the tangent vector \( \overrightarrow{v_{C_{\theta_0}}} \) at \( M_{00} \) define a characteristic circle of the future Dupin cyclide. We have the same remark if we replace \( M_{01} \) by \( M_{10} \) and \( \overrightarrow{v_{C_{\theta_0}}} \) by \( \overrightarrow{v_{C_{\psi_0}}} \).
Theorem 2 permits to compute, from two pencils of spheres with a base circle, a sphere in each pencil, such that these spheres are tangent. In this theorem, the four spheres $\sigma_\theta, \sigma_\psi, \sigma^\perp_\theta$ and $\sigma^\perp_\psi$ verify

$$
\begin{align*}
\mathcal{L}_{4,1} \left( \overrightarrow{O5\sigma_\psi}, \overrightarrow{O5\sigma_\theta} \right) \times \mathcal{L}_{4,1} \left( \overrightarrow{O5\sigma_\psi}, \overrightarrow{O5\sigma^\perp_\theta} \right) & \leq 0 \\
\mathcal{L}_{4,1} \left( \overrightarrow{O5\sigma_\theta}, \overrightarrow{O5\sigma^\perp_\psi} \right) \times \mathcal{L}_{4,1} \left( \overrightarrow{O5\sigma^\perp_\psi}, \overrightarrow{O5\sigma^\perp_\theta} \right) & \leq 0
\end{align*}
$$

(3.9)

**Theorem 2** Let $C_\theta$ and $C_\psi$ be two circles passing through the point $M_{00}$ such that the tangent vectors to these circles at $M_{00}$ are perpendicular.

Let $\sigma_\theta$ (respectively $\sigma_\psi$) be the point on $\Lambda^4$ corresponding to an oriented sphere having $C_\theta$ (respectively $C_\psi$) as great circle.

Let $\sigma^\perp_\theta$ (respectively $\sigma^\perp_\psi$) be the point on $\Lambda^4$ corresponding to an oriented plane containing the circle $C_\theta$ (respectively $C_\psi$).

On $\Lambda^4$, the parameterizations $\theta_0 \mapsto \gamma_\theta (\theta_0)$ and $\psi_0 \mapsto \gamma_\psi (\psi_0)$ of the circles which represent these two pencils of spheres, are given by the Formula (3.8).

Let $A$ and $B$ be

$$
A = \sqrt{\mathcal{L}_{4,1} \left( \overrightarrow{O5\sigma_\psi}, \overrightarrow{O5\sigma_\theta} \right)^2 + \mathcal{L}_{4,1} \left( \overrightarrow{O5\sigma_\psi}, \overrightarrow{O5\sigma^\perp_\theta} \right)^2}
$$

$$
B = \sqrt{\mathcal{L}_{4,1} \left( \overrightarrow{O5\sigma_\theta}, \overrightarrow{O5\sigma^\perp_\psi} \right)^2 + \mathcal{L}_{4,1} \left( \overrightarrow{O5\sigma^\perp_\psi}, \overrightarrow{O5\sigma^\perp_\theta} \right)^2}
$$

(3.10)

Let $\theta_s$ be the solution of the system

$$
\begin{align*}
\cos (\theta_s) &= A^2 - B^2 \\
\sin (\theta_s) &= 2 A B
\end{align*}
$$

Let $\theta_0 = -\frac{\theta_s}{2}$ or $\theta_0 = -\frac{\theta_s}{2} + \pi$.

Let $a_{\theta_0}$ and $b_{\theta_0}$ be

$$
a_{\theta_0} = \cos (\theta_0) \mathcal{L}_{4,1} \left( \overrightarrow{O5\sigma_\theta}, \overrightarrow{O5\sigma_\psi} \right) + \sin (\theta_0) \mathcal{L}_{4,1} \left( \overrightarrow{O5\sigma_\psi}, \overrightarrow{O5\sigma^\perp_\theta} \right)
$$

$$
b_{\theta_0} = \cos (\theta_0) \mathcal{L}_{4,1} \left( \overrightarrow{O5\sigma_\theta}, \overrightarrow{O5\sigma^\perp_\psi} \right) + \sin (\theta_0) \mathcal{L}_{4,1} \left( \overrightarrow{O5\sigma^\perp_\psi}, \overrightarrow{O5\sigma^\perp_\theta} \right)
$$

(3.11)

Let $\psi_0$ be the solution of the system

$$
\begin{align*}
\cos (\psi_0) &= a_{\theta_0} \\
\sin (\psi_0) &= b_{\theta_0}
\end{align*}
$$

(3.12)
Then, in $\mathcal{E}_3$, the spheres corresponding to the points $\gamma_\theta (\theta_0)$ and $\gamma_\psi (\psi_0)$ are tangent at $M_{00}$.

Proof: Appendix 5.

We can remark that the spheres $\gamma_\theta (\theta_0)$ and $\gamma_\theta (\theta_0 + \pi)$ represent the same geometric sphere in $\mathcal{E}_3$.

From a pencil of tangent spheres at a point, Lemma 3.1 permits to determine the sphere of this pencil passing through another point. This lemma will be useful in the steps (12) to (14) of Algorithm 1.

**Lemma 3.1**:

Let $\sigma_0$ (respectively $\overrightarrow{m_{00}}$) be the representation of an oriented sphere (respectively a point $M_{00}$) of $\mathcal{E}_3$.

Let $l_{00} : t \mapsto \sigma_0 + t \overrightarrow{m_{00}}$ be a parameterization of a light-like line (on $\Lambda^4$) which represents the spheres tangent to $\sigma_0$ at $M_{00}$.

Let $\overrightarrow{m_{10}}$ be a light-like vector which represents the point $M_{10}$ of $\mathcal{E}_3$.

The point $M_{10}$ belongs to the sphere defined by the point $l_{00} (t_0)$ if and only if

$$ t_0 = f (\sigma_0, \overrightarrow{m_{00}}, \overrightarrow{m_{10}}) = - \frac{\mathcal{L}_{4,1} (\overrightarrow{O_5 \sigma_0}, \overrightarrow{m_{10}})}{\mathcal{L}_{4,1} (\overrightarrow{m_{00}}, \overrightarrow{m_{10}})} \quad (3.13) $$

Proof:

By construction of the space of spheres, the point $M_{10}$ belongs to the sphere defined by the point $l_{00} (t_0)$ if and only if (an analytic proof is easy):

$$ \mathcal{L}_{4,1} (\overrightarrow{O_5 l_{00} (t_0)}, \overrightarrow{m_{10}}) = 0 \quad (3.14) $$

and we have

$$ \mathcal{L}_{4,1} (\overrightarrow{O_5 l_{00} (t_0)}, \overrightarrow{m_{10}}) = 0 \iff \mathcal{L}_{4,1} (\overrightarrow{O_5 \sigma_0} + t_0 \overrightarrow{m_{00}}, \overrightarrow{m_{10}}) = 0 $$

$$ \iff \mathcal{L}_{4,1} (\overrightarrow{O_5 \sigma_0}, \overrightarrow{m_{10}}) + t_0 \mathcal{L}_{4,1} (\overrightarrow{m_{00}}, \overrightarrow{m_{10}}) = 0 $$

$$ \iff t_0 = - \frac{\mathcal{L}_{4,1} (\overrightarrow{O_5 \sigma_0}, \overrightarrow{m_{10}})}{\mathcal{L}_{4,1} (\overrightarrow{m_{00}}, \overrightarrow{m_{10}})} $$
Algorithm 1 Computation of a Dupin cyclide principal patch in the space of spheres

**Input:** Four points \((M_{ij})_{(i,j) \in [0,1]^2}\) on a circle and two perpendicular vectors \(\vec{v}_{C_{\theta_0}}\) and \(\vec{v}_{C_{\psi_0}}\).

1. **For** \(i \text{ from } 0 \text{ to } 1\) **do**
   **For** \(j \text{ from } 0 \text{ to } 1\) **do**
   Computation of \(\vec{m}_{ij}\) which represents the point \(M_{ij}\) on the paraboloid \(P\)
   **End do**
   **End do**.

2. Construction of the circle \(C_{\theta}\) passing through the points \(M_{00}\) and \(M_{01}\) such that the tangent vector at \(M_{00}\) is \(\vec{v}_{C_{\theta_0}}\).

3. Construction of the circle \(C_{\psi}\) passing through the points \(M_{00}\) and \(M_{10}\) such that the tangent vector at \(M_{00}\) is \(\vec{v}_{C_{\psi_0}}\).

4. Construction of \(\sigma_{\theta_i}\) which is the representation of an oriented sphere in \(E^3\) having \(C_{\theta}\) as great circle.

5. Construction of \(\sigma_{\theta_0}^\perp\) which is the representation of an oriented plane in \(E^3\) containing \(C_{\theta}\).

6. Construction of \(\sigma_{\psi_i}\) which is the representation an oriented sphere in \(E^3\) having \(C_{\psi}\) as great circle.

7. Construction of \(\sigma_{\psi_0}^\perp\) which is the representation of an oriented plane in \(E^3\) containing \(C_{\psi}\).

8. Determination of the circle \(\gamma_{\theta_i} (\theta_0)\) representing the pencil of spheres with base circle \(C_{\theta}\), Formula (3.8).

9. Determination of the circle \(\gamma_{\psi_i} (\psi_0)\) representing the pencil of spheres with base circle \(C_{\psi}\), Formula (3.8).

10. Computation of \(\theta_0\) and \(\psi_0\) using Theorem 2.

11. Computation of the point \(\sigma_{\theta_0} = \gamma_{\theta} (\theta_0)\) and \(\sigma_{\psi_0} = \gamma_{\psi} (\psi_0)\).

12. Determination of a parameterization of the light-like line \(l_{01} : t \mapsto \sigma_{\psi_0} + t \vec{m}_{10}\).

13. Computation of \(t_1 = f (\sigma_{\psi_0}, \vec{m}_{10}, \vec{m}_{11})\) using Formula (3.13).

14. Computation of the point \(\sigma_{\theta_2} = l_{01} (t_1)\).

15. Computation of the point \(\dot{\sigma}_{\theta_0}\), Formula (2.22).

**Output:** Three points \(\sigma_{\theta_0}, \sigma_{\theta_2}\) and \(\dot{\sigma}_{\theta_0}\) on \(\Lambda^4\) which generate one of the two planes defining a Dupin cyclide.
Figure 9. Construction of a Dupin cyclide principal patch in the space of spheres $\Lambda^4$, the lines $(\sigma_{\theta_0}\sigma_{\psi_0})$, $(\sigma_{\theta_2}\sigma_{\psi_0})$, $(\sigma_{\theta_0}\sigma_{\psi_2})$, and $(\sigma_{\theta_0}\sigma_{\psi_0})$, are light-like lines and have $m_{00}$, $m_{10}$, $m_{11}$, and $m_{01}$ as direction vectors respectively: The pink (respectively green) 2-plane is the plane containing the pencil of spheres with base circle $C_{\theta_0}$ (respectively $C_{\psi_0}$): We use Theorem 2 to compute the spheres $\sigma_{\theta_0}$ and $\sigma_{\psi_0}$ whereas we use Lemma 3.1 to compute the spheres $\sigma_{\theta_2}$ and $\sigma_{\psi_2}$: Each pencil of spheres is represented by a circle (for the Lorentz form) which is the intersection between $\Lambda^4$ and an affine space-like 2-plane passing through $O_5$.
Figure 10. Construction of a Dupin cyclide principal patch in the space of spheres $\Lambda^4$, the pink 2-plane contains the circle representing the pencil of spheres with a base the circle $C_\theta$: The green 2-plane contains the circle representing a family of spheres which defines the Dupin cyclide: This plane is generated by the sphere $\sigma_{\theta_0}$ and the characteristic circle $\sigma_{\theta_0} \cap \sigma_{\theta_0}$ which is defined by the pair $(\sigma_{\theta_0}, \gamma_0'(\theta_0))$, these three elements are computed using Algorithm 1.
For simplicity, sometimes, we use the same notation for a sphere in $E_3$ and its representation on $\Lambda^4$.

In Algorithm 1, we propose a method to compute the Dupin cyclide. To obtain the required principal patch, it is enough to trim the previous Dupin cyclide and to keep the useful part. The steps (1) to (7) permit to transfer, on $\Lambda^4$, the constraints enforced in $E_3$. This algorithm uses Theorem 2, the spheres $\sigma_{\theta_1}, \sigma_{\theta_1}^\perp, \sigma_{\psi_1}$, and $\sigma_{\psi_1}^\perp$ verify the conditions given by the Formula (3.9).

The steps (8) and (9) permit to obtain the usual parameterization of a circle in an affine 2-plane equipped with a dot product. The steps (10) and (11) permit to determine the two spheres tangent at the point $M_{00}$, each sphere belongs to each pencil. The steps (12) to (14) permit to compute the second sphere $\sigma_{\theta_2}$. Using Lemma 3.1, we compute the sphere $\sigma_{\theta_2}$, tangent to $S_{\psi_0}$ at $M_{10}$, passing through the point $M_{11}$.

In the step (15), we compute the last sphere $\sigma_{\theta_0}$ defined by the sphere $\sigma_{\theta_0}$ and the tangent vector $\vec{\gamma_0}'(\theta_0)$, Formula (2.22).

We can remark that iterating the part of the algorithm corresponding to the steps (12) to (14) computes the sphere $S_{\psi_2}$ of the second family: the second circle on $\Lambda^4$ representing the same Dupin cyclide would be defined by the intersection between $\Lambda^4$ and the affine 2-plane generated by $\sigma_{\psi_2}, \sigma_{\psi_0}$, and $\sigma_{\psi_0}^\perp$. Continuing the process, we would compute the point $\sigma_{\theta_3}$ which represents the oriented sphere $S_{\theta_3}$ in $E_3$. Since we use Theorem 2 and Algorithm 1, we would have $\sigma_{\theta_3} = \sigma_{\theta_0}$, generally, this result is wrong and $\sigma_{\theta_3}$ belongs to the curve $\gamma_0$ but is distinct to $\sigma_{\theta_0}$.

Figure 9 shows the two pencils of spheres, each pencil is represented by a circle with centre $O_5$ and belongs to a space-like 2-plane passing through $O_5$. The points $\sigma_{\theta_0}$ and $\sigma_{\psi_0}$ which belong to the two pencils, are determined using Theorem 2. The points $\sigma_{\theta_0}$ (respectively $\sigma_{\psi_0}$) is the representation of the oriented sphere $S_{\theta_0}$ (respectively $S_{\psi_0}$) in $E_3$. The computation of the point $\sigma_{\theta_2}$ is made using Lemma 3.1: $\sigma_{\theta_2}$ belongs to a light-like line and we know $\vec{m}_{11}$ which represents, in $E_3$, the point $M_{11}$ belonging to the sphere $S_{\theta_2}$ represented by $\sigma_{\theta_2}$. The spheres $S_{\theta_0}$ and $S_{\psi_0}$ are tangent at $M_{00}$. The spheres $S_{\theta_2}$ and $S_{\psi_0}$ are tangent at $M_{10}$.

The spheres $S_{\theta_2}$ and $S_{\psi_2}$ would be tangent at $M_{11}$. Finally, The spheres $S_{\psi_2}$ and $S_{\theta_0}$ would be tangent at $M_{01}$.

Figure 10 shows the point $\sigma_{\theta_0}$ which belongs to the circle $\gamma_0$. The characteristic circle on $\sigma_{\theta_0}$ is determined by the pair $\left(\sigma_{\theta_0}, \vec{\gamma_0}'(\theta_0)\right)$ where $\vec{\gamma_0}'(\theta_0)$ is the tangent vector to the curve $\gamma_0$ at $\sigma_{\theta_0}$. Thus, the Dupin cyclide is completely defined: it is the intersection between $\Lambda^4$ and the affine 2-plane generated by the points $\sigma_{\theta_0}$ and $\sigma_{\theta_2}$ and the vector $\vec{\gamma_0}'(\theta_0)$). This vector determines the point.
\( \sigma_{\theta_0} \) which represents, in \( \mathcal{E}_3 \), the sphere \( S_{\theta_0} \) orthogonal to the sphere \( S_{\theta_0} \).

### 3.5 Example

In this section, we introduce the following notation: \( C(\Omega, r) \) (respectively \( S(\Omega, r) \)) is a circle (respectively sphere) with centre \( \Omega \) and with radius \( r \).

The four vertices of the future principal patch \( M_{00} = (2 \sqrt{2}, 2 \sqrt{2}, 0), M_{10} = (2 \sqrt{3}, -2, 0), M_{01} = (-2, 2 \sqrt{3}, 0) \) and \( M_{11} = (-2 \sqrt{2}, -2 \sqrt{2}, 0) \) belong to the circle \( C(O, 4) \) (in black) in the plane \( \mathcal{P}_z : z = 0 \), Figure 11(a).

The characteristic circles are modelized by two couples of rational quadratic Bézier curves defined by the weighted control points \( (M_{00}, 1), (P_{01}, \pm \omega_{01}), (M_{01}, 1) \) and \( (M_{00}, 1), (P_{10}, \pm \omega_{10}), (M_{10}, 1) \) with

\[
(P_{01}, \omega_{01}) \simeq \left(-0.129, -0.980, 2\right), 0.466
\]

and

\[
(P_{10}, \omega_{10}) \simeq \left(3.798, \frac{1}{2}, -3\right), 0.621
\]

The initial spheres are characterized by the centres and the radii, noted \( (\Omega, r) \), and their representations on \( \Lambda^4 \) are given in the Table 8.

We obtain

\[
(\theta_0, \theta_0, \psi_0) \simeq (0.7470776248, -0.3735388124, 2.164840770)
\]

and the spheres which define the Dupin cyclide are given in the Table 9. Since we have

\[
O_5 \overrightarrow{\sigma_{\theta_0}} = \cos(\theta_0) \ O_5 \overrightarrow{\sigma_{\theta_1}} + \sin(\theta_0) \ O_5 \overrightarrow{\sigma_{\theta_1}}
\]

the sphere \( \sigma_{\theta_0} \) is computed using the following formula

\[
O_5 \overrightarrow{\sigma_{\theta_0}} = -\sin(\theta_0) \ O_5 \overrightarrow{\sigma_{\theta_1}} + \cos(\theta_0) \ O_5 \overrightarrow{\sigma_{\theta_1}}
\]
Figure 11. Example: (a) the vertices $M_{00}$, $M_{01}$, $M_{11}$ and $M_{10}$ belong to the black circle in the plane $P_z : z = 0$, the other circle are the characteristic circle to the Dupin cylide to compute, (b) The Dupin cylide is characterized, on $\Lambda^4$, by the three points $\sigma_{\theta_0}$, $\sigma_{\theta_2}$ and $\sigma_{\hat{\theta}_0}$ which represent, in $E^3$, the three spheres $S_{\theta_0}$, $S_{\theta_2}$ and $S_{\hat{\theta}_0}$: We can note the two spheres $S_{\theta_0}$ and $S_{\hat{\theta}_0}$ are orthogonal
and one of two affine 2-planes which characterize the Dupin cyclide is generated by the points $\sigma_{\theta_0}$ and $\sigma_{\theta_2}$ and the vector $\overrightarrow{O_5\sigma_{\theta_0}}$.

We obtain the Dupin cyclide parameters

$$(a, c, \mu) \simeq (4.934, 1.803, 2.849)$$

and using the Formula (2.6) and the following affine matrix (see [14]),

$$
\begin{pmatrix}
0.059 & 0.039 & 0.998 & 0.698 \\
0.379 & -0.925 & 0.013 & -0.301 \\
0.923 & 0.377 & -0.069 & 0.471 \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

(3.15)

we can place the Dupin cyclide in the scene, Figure 12. Then, we can calculate the bounds of the Dupin cyclide principal patch, Figure 13. We obtain $\theta_0 \simeq 2.521616545$, $\theta_1 \simeq 4.652818777$, $\psi_0 \simeq -0.9430762257$ and $\psi_1 \simeq 1.266085740$.

## 4 Conclusions

In this paper, we have given an algorithm to compute a principal patch Dupin cyclide in the space of spheres. The constraints are four vertices on a circle and two perpendicular tangents to the patch at one of the vertices. The use of the space of spheres permits to simplify the work made by Garnier and Gentil: we never solve any equations.

In the future, we should compute a Dupin cyclide triangle tangent to three planes at one point on each previous plane. We expect to build some subdivision schema with Dupin cyclide patches and Dupin cyclide triangles.
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References


Figure 13. Two views of the Dupin cyclide principal patch obtained from Figure 8 using Algorithm 1.
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5 Appendix I: proof of theorem 2:

Before the proof, we need two lemmas.

5.1 Lemmas

Lemma 1 First, we have

\[
\mathcal{L}_{4,1}(\overrightarrow{O_5\sigma_{\psi_0}}, \overrightarrow{O_5\sigma_{\theta_0}}) \times \mathcal{L}_{4,1}(\overrightarrow{O_5\sigma_{\theta_0}}, \overrightarrow{O_5\sigma_{\psi_0}}) = 
\]

(5.1)

and

\[
\mathcal{L}_{4,1}(\overrightarrow{O_5\sigma_{\psi_0}}, \overrightarrow{O_5\sigma_{\theta_0}})^2 + \mathcal{L}_{4,1}(\overrightarrow{O_5\sigma_{\theta_0}}, \overrightarrow{O_5\sigma_{\psi_0}})^2 + 
\]

\[
\mathcal{L}_{4,1}(\overrightarrow{O_5\sigma_{\theta_0}}, \overrightarrow{O_5\sigma_{\psi_0}})^2 + \mathcal{L}_{4,1}(\overrightarrow{O_5\sigma_{\theta_0}}, \overrightarrow{O_5\sigma_{\psi_0}})^2 = 1
\]

Second, we have

\[(A \cos \theta_0 - B \sin \theta_0)^2 = 1 \text{ and } a_{\theta_0}^2 + b_{\theta_0}^2 = 1 \] (5.3)

where A and B are given by Formula (3.10) whereas \(a_{\theta_0}\) and \(b_{\theta_0}\) are given by Formula (3.11).

Proof:

Without loss of generality, we can always consider the following case.

The point \(M_{00}(0, 1, 0)\) belongs to two circles. Let \(\Omega_{\theta_0}(0, 0, 0)\) (respectively \(r_{\theta_0} = 1\)) be the centre (respectively the radius) of the first circle \(C_{\theta_0}\) (and of the first sphere). Let \(\vec{v}_{C_{\theta_0}}(-1, 0, 0)\) be a tangent vector to the circle at \(M_{00}\). Using Formulae (2.16) and (2.18), we have

\[
\sigma_{\theta_0} = (0, 0, 0, -1) \quad \text{and} \quad \sigma_{\theta_0}^{\perp} = (0, 0, 0, -1, 0)
\]

Let \(\vec{v}_{C_{\theta_0}}(0, \alpha, \beta)\) be an unit vector orthogonal to \(\vec{v}_{C_{\theta_0}}\). We have to distinguish two cases.

5.1.1 First case: \(\alpha = 1 \Rightarrow \beta = 0\).

The centre of the second circle \(C_{\psi_0}\) is

\[\Omega_{\psi_0}(x_0, 1, z_0)\]
and the radius of \( C_{\psi_0} \) is

\[ r_{\psi_0} = \sqrt{x_0^2 + z_0^2} \]

Thus, the sphere \( S(\Omega_{\psi_0}, r_{\psi_0}) \) is represented on \( \Lambda^4 \) by

\[
\sigma_{\psi} = \begin{pmatrix}
\frac{1}{\sqrt{x_0^2 + z_0^2}}, & \frac{x_0}{\sqrt{x_0^2 + z_0^2}}, & \frac{1}{\sqrt{x_0^2 + z_0^2}}, & \frac{z_0}{\sqrt{x_0^2 + z_0^2}}, & 0
\end{pmatrix}
\]

whereas the plane containing the circle \( C_{\theta_0} \), orthogonal to the previous sphere, is represented on \( \Lambda^4 \) by

\[
\sigma_{\psi}^\perp = \begin{pmatrix}
0, & \frac{z_0}{\sqrt{x_0^2 + z_0^2}}, & 0, & -\frac{x_0}{\sqrt{x_0^2 + z_0^2}}, & 0
\end{pmatrix}
\]

We have

\[
L_{4,1}(\overrightarrow{O_5\sigma_{\theta}}, \overrightarrow{O_5\sigma_{\psi}}) \times L_{4,1}(\overrightarrow{O_5\sigma_{\psi}}, \overrightarrow{O_5\sigma_{\theta}^\perp}) = -\frac{z_0}{\sqrt{x_0^2 + z_0^2}} \times 0 \leq 0
\]

and

\[
L_{4,1}(\overrightarrow{O_5\sigma_{\theta}}, \overrightarrow{O_5\sigma_{\psi}^\perp}) \times L_{4,1}(\overrightarrow{O_5\sigma_{\psi}^\perp}, \overrightarrow{O_5\sigma_{\theta}^\perp}) = \frac{x_0}{\sqrt{x_0^2 + z_0^2}} \times 0 \leq 0
\]

and the conditions given in Formula (3.9) are true.

Then, the Formulae (5.1) and (5.2) are true.

As we have

\[
L_{4,1}(\overrightarrow{O_5\sigma_{\theta}}, \overrightarrow{O_5\sigma_{\psi}}) = L_{4,1}(\overrightarrow{O_5\sigma_{\theta}}, \overrightarrow{O_5\sigma_{\psi}^\perp})
\]

the condition given in Formula (3.9) is true.

Moreover, we have \( A = 0 \) and \( B = 1 \). Thus, we obtain \( \theta_s = \pi \). Choosing \( \theta_0 = \frac{\pi}{2} \) and using Formula (3.11), we obtain

\[
\begin{aligned}
a_{\theta_0} &= -\frac{z_0}{\sqrt{x_0^2 + z_0^2}} \\
b_{\theta_0} &= \frac{x_0}{\sqrt{x_0^2 + z_0^2}}
\end{aligned}
\]

and we have \( a_{\theta_0}^2 + b_{\theta_0}^2 = 1 \). We have too

\[
(A \cos \theta_0 - B \sin \theta_0)^2 = \left(0 \times \cos \frac{\pi}{2} - 1 \times \sin \frac{\pi}{2}\right)^2 = 1
\]

\[\square\]
5.1.2 Second case: $0 \leq \alpha < 1$ and $\beta = \sqrt{1 - \alpha^2}$.

Without loss of generality, we can always consider the following case.

The centre of the second circle $C_{\psi_0}$ is

$$\Omega_{\psi_0} \left( x_0, y_0, -\frac{\alpha (y_0 - 1)}{\sqrt{1 - \alpha^2}} \right)$$

whereas its radius is

$$r_{\psi_0} = \sqrt{\frac{x_0^2 (1 - \alpha^2) + (y_0 - 1)^2}{1 - \alpha^2}}$$

Thus, we have on the one hand

$$\sigma_{\psi_I} = \frac{1}{\sqrt{(1 - \alpha^2) x_0^2 + (y_0 - 1)^2}} \Sigma_{\psi_I}$$

where

$$\Sigma_{\psi_I} = \left( y_0 \sqrt{1 - \alpha^2} ; x_0 \sqrt{1 - \alpha^2} ; y_0 \sqrt{1 - \alpha^2} ; (1 - y_0) \alpha ; (y_0 - 1) \sqrt{1 - \alpha^2} \right)$$

and we have on the other hand

$$\sigma_{\psi_I}^\perp = \frac{1}{\sqrt{(1 - \alpha^2) x_0^2 + (y_0 - 1)^2}} \Sigma_{\psi_I}^\perp$$

where

$$\Sigma_{\psi_I}^\perp = \left( x_0 (1 - \alpha^2) , 1 - y_0 , x_0 (1 - \alpha^2) , -x_0 \alpha \sqrt{1 - \alpha^2} , x_0 (1 - \alpha^2) \right)$$

We can remark the condition given in Formula (3.9) is true

$$\mathcal{L}_{4,1} \left( \overrightarrow{O_5 \sigma_{\psi_I}}, \overrightarrow{O_5 \sigma_{\theta_I}} \right) \times \mathcal{L}_{4,1} \left( \overrightarrow{O_5 \sigma_{\psi_I}}, \overrightarrow{O_5 \sigma_{\psi_I}^\perp} \right) = -\frac{\sqrt{1 - \alpha^2} \alpha (y_0 - 1)^2}{(1 - \alpha^2) x_0^2 + (y_0 - 1)^2} = 0$$

and

$$\mathcal{L}_{4,1} \left( \overrightarrow{O_5 \sigma_{\theta_I}}, \overrightarrow{O_5 \sigma_{\psi_I}^\perp} \right) \times \mathcal{L}_{4,1} \left( \overrightarrow{O_5 \sigma_{\psi_I}^\perp}, \overrightarrow{O_5 \sigma_{\psi_I}^\perp} \right) = \frac{x_0^2 (\alpha^2 - 1) \alpha \sqrt{1 - \alpha^2}}{(1 - \alpha^2) x_0^2 + (y_0 - 1)^2} \leq 0$$

Moreover

$$\mathcal{L}_{4,1} \left( \overrightarrow{O_5 \sigma_{\theta_I}}, \overrightarrow{O_5 \sigma_{\psi_I}^\perp} \right) \times \mathcal{L}_{4,1} \left( \overrightarrow{O_5 \sigma_{\psi_I}^\perp}, \overrightarrow{O_5 \sigma_{\psi_I}^\perp} \right) = \frac{(1 - y_0) x_0 \alpha (1 - \alpha^2)}{(1 - \alpha^2) x_0^2 + (y_0 - 1)^2} =$$

$$\mathcal{L}_{4,1} \left( \overrightarrow{O_5 \sigma_{\theta_I}}, \overrightarrow{O_5 \sigma_{\psi_I}^\perp} \right) \times \mathcal{L}_{4,1} \left( \overrightarrow{O_5 \sigma_{\psi_I}}, \overrightarrow{O_5 \sigma_{\theta_I}^\perp} \right)$$

and the Formula (5.1) is true. Thus
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\[ A^2 = \mathcal{L}_{4,1} \left( \overrightarrow{O_5 \sigma_{\theta_1}}, \overrightarrow{O_5 \sigma_{\psi}} \right)^2 + \mathcal{L}_{4,1} \left( \overrightarrow{O_5 \sigma_{\psi}}, \overrightarrow{O_5 \sigma_{\theta_1}} \right)^2 \]

\[ = \frac{(1 - y_0)^2 (1 - \alpha^2) + x_0^2 (\alpha^2 - 1)^2}{(1 - \alpha^2)x_0^2 + (y_0 - 1)^2} \]

\[ = \frac{(1 - \alpha^2) (1 - y_0)^2 + x_0^2 (1 - \alpha^2)}{(1 - \alpha^2)x_0^2 + (y_0 - 1)^2} \]

\[ = 1 - \alpha^2 \]

\[ B^2 = \mathcal{L}_{4,1} \left( \overrightarrow{O_5 \sigma_{\psi}}, \overrightarrow{O_5 \sigma_{\theta_1}} \right)^2 + \mathcal{L}_{4,1} \left( \overrightarrow{O_5 \sigma_{\theta_1}}, \overrightarrow{O_5 \sigma_{\psi}} \right)^2 \]

\[ = \frac{\alpha^2 (y_0 - 1)^2 + x_0^2 \alpha^2 (1 - \alpha^2)}{(1 - \alpha^2)x_0^2 + (y_0 - 1)^2} \]

\[ = \frac{\alpha^2 (y_0 - 1)^2 + x_0^2 (1 - \alpha^2)}{(1 - \alpha^2)x_0^2 + (y_0 - 1)^2} \]

\[ = \alpha^2 \]

Thus, \( A^2 + B^2 = 1 \) and the Formula (5.2) is true. Moreover, \( \theta_s = \pi \) is solution of the system

\[
\begin{align*}
\cos (\theta_s) &= 1 - 2 \alpha^2 \\
\sin (\theta_s) &= 2 \alpha \sqrt{1 - \alpha^2}
\end{align*}
\]

and using the relation \( 2 \cos^2 (\theta_0) = 1 + \cos (2 \theta_0) = 1 + \cos (-\theta_s), \theta_0 \) verifies

\[
\begin{align*}
\cos^2 (\theta_0) &= 1 - \alpha^2 \\
\sin^2 (\theta_0) &= \alpha^2
\end{align*}
\]

and we obtain

\[
\begin{align*}
\cos (\theta_0) &= \varepsilon_c \sqrt{1 - \alpha^2} \\
\sin (\theta_0) &= \varepsilon_s \alpha, \quad (\varepsilon_c, \varepsilon_s) \in \{-1, 1\}
\end{align*}
\]

From the following formula

\[
\frac{2 \tan (\theta_0)}{1 - \tan^2 (\theta_0)} = \tan (2 \theta_0) = -\tan (\theta_s)
\]
we have
\[
\frac{2\varepsilon_s \alpha}{\varepsilon_c \sqrt{1 - \alpha^2}} \times \frac{1}{1 - \frac{\alpha^2}{1 - \alpha^2}} = -\frac{2 \alpha}{1 - 2 \alpha^2} \sqrt{1 - \alpha^2}
\]
which can be simplified into
\[
\frac{\varepsilon_s}{\varepsilon_c \sqrt{1 - \alpha^2}} \times \frac{1 - \alpha^2}{1 - 2 \alpha^2} = -\frac{1}{1 - 2 \alpha^2} \sqrt{1 - \alpha^2}
\]
and then, we obtain
\[
\varepsilon_s = -\varepsilon_c
\]
We can remark we have the following relation
\[
(A \cos \theta_0 - B \sin \theta_0)^2 = \left(\sqrt{1 - \alpha^2} \varepsilon_c \sqrt{1 - \alpha^2} - \alpha \varepsilon_r \alpha\right)^2 = \left(\varepsilon_c \left(1 - \alpha^2 + \alpha^2\right)\right) = 1
\]
(5.6)
and then, we obtain
\[
\begin{align*}
a_{\theta_0} &= \frac{\varepsilon_c \sqrt{1 - \alpha^2} (1 - y_0) \sqrt{1 - \alpha^2} - \varepsilon_c \alpha^2 (y_0 - 1)}{\sqrt{(1 - \alpha^2) x_0^2 + (y_0 - 1)^2}} \\
b_{\theta_0} &= \frac{\varepsilon_c \sqrt{1 - \alpha^2} x_0 (\alpha^2 - 1) - \varepsilon_c x_0 \alpha \sqrt{1 - \alpha^2}}{\sqrt{(1 - \alpha^2) x_0^2 + (y_0 - 1)^2}}
\end{align*}
\]
which is equivalent to
\[
\begin{align*}
a_{\theta_0} &= (y_0 - 1) \frac{-\varepsilon_c (1 - \alpha^2) - \varepsilon_c \alpha^2}{\sqrt{(1 - \alpha^2) x_0^2 + (y_0 - 1)^2}} \\
b_{\theta_0} &= x_0 \sqrt{1 - \alpha^2} \frac{\varepsilon_c (\alpha^2 - 1) - \varepsilon_c \alpha^2}{\sqrt{(1 - \alpha^2) x_0^2 + (y_0 - 1)^2}}
\end{align*}
\]
which is equivalent to
\[
\begin{align*}
a_{\theta_0} &= \frac{-\varepsilon_c (y_0 - 1)}{\sqrt{(1 - \alpha^2) x_0^2 + (y_0 - 1)^2}} \\
b_{\theta_0} &= \frac{-\varepsilon_c x_0 \sqrt{1 - \alpha^2}}{\sqrt{(1 - \alpha^2) x_0^2 + (y_0 - 1)^2}}
\end{align*}
(5.7)
and, finally, we obtain
\[
a_{\theta_0}^2 + b_{\theta_0}^2 = 1
\]
(5.8)
Lemma 2: We have

\[
\sqrt{\mathcal{L}_{4,1} \left( \overrightarrow{O_5 \sigma_{\psi_1}}, \overrightarrow{O_5 \sigma_{\theta_1}} \right)^2 + \mathcal{L}_{4,1} \left( \overrightarrow{O_5 \sigma_{\theta_1}}, \overrightarrow{O_5 \sigma_{\psi_1}} \right)^2} \times
\]
\[
\sqrt{\mathcal{L}_{4,1} \left( \overrightarrow{O_5 \sigma_{\psi_1}}, \overrightarrow{O_5 \sigma_{\theta_1}} \right)^2 + \mathcal{L}_{4,1} \left( \overrightarrow{O_5 \sigma_{\theta_1}}, \overrightarrow{O_5 \sigma_{\psi_1}} \right)^2 + \mathcal{L}_{4,1} \left( \overrightarrow{O_5 \sigma_{\psi_1}}, \overrightarrow{O_5 \sigma_{\psi_1}} \right) \mathcal{L}_{4,1} \left( \overrightarrow{O_5 \sigma_{\theta_1}}, \overrightarrow{O_5 \sigma_{\psi_1}} \right) \mathcal{L}_{4,1} \left( \overrightarrow{O_5 \sigma_{\psi_1}}, \overrightarrow{O_5 \sigma_{\psi_1}} \right) = 0
\]

(5.9)

Proof:
Let us recall the conditions given in Formula (3.9). Let us suppose Formula (5.9) is true, then, we have

\[
\left( \mathcal{L}_{4,1} \left( \overrightarrow{O_5 \sigma_{\psi_1}}, \overrightarrow{O_5 \sigma_{\theta_1}} \right)^2 + \mathcal{L}_{4,1} \left( \overrightarrow{O_5 \sigma_{\theta_1}}, \overrightarrow{O_5 \sigma_{\psi_1}} \right)^2 \right) \times
\]
\[
\left( \mathcal{L}_{4,1} \left( \overrightarrow{O_5 \sigma_{\psi_1}}, \overrightarrow{O_5 \sigma_{\theta_1}} \right)^2 + \mathcal{L}_{4,1} \left( \overrightarrow{O_5 \sigma_{\theta_1}}, \overrightarrow{O_5 \sigma_{\psi_1}} \right)^2 \right) =
\]
\[
\left( \mathcal{L}_{4,1} \left( \overrightarrow{O_5 \sigma_{\psi_1}}, \overrightarrow{O_5 \sigma_{\theta_1}} \right) \mathcal{L}_{4,1} \left( \overrightarrow{O_5 \sigma_{\psi_1}}, \overrightarrow{O_5 \sigma_{\psi_1}} \right) - \mathcal{L}_{4,1} \left( \overrightarrow{O_5 \sigma_{\theta_1}}, \overrightarrow{O_5 \sigma_{\psi_1}} \right) \mathcal{L}_{4,1} \left( \overrightarrow{O_5 \sigma_{\psi_1}}, \overrightarrow{O_5 \sigma_{\psi_1}} \right) \right)^2
\]

which is equivalent (after some computations) to

\[
\left( \mathcal{L}_{4,1} \left( \overrightarrow{O_5 \sigma_{\psi_1}}, \overrightarrow{O_5 \sigma_{\theta_1}} \right) \mathcal{L}_{4,1} \left( \overrightarrow{O_5 \sigma_{\psi_1}}, \overrightarrow{O_5 \sigma_{\psi_1}} \right) - \mathcal{L}_{4,1} \left( \overrightarrow{O_5 \sigma_{\theta_1}}, \overrightarrow{O_5 \sigma_{\psi_1}} \right) \mathcal{L}_{4,1} \left( \overrightarrow{O_5 \sigma_{\psi_1}}, \overrightarrow{O_5 \sigma_{\psi_1}} \right) \right)^2 = 0
\]

and this relation is true from Formula (5.1).

5.2 Proof of Theorem 2:

From Formula (3.8), we have

\[
\overrightarrow{O_5 \gamma_0 (\theta_0)} = \cos (\theta_0) \overrightarrow{O_5 \sigma_{\theta_1}} + \sin (\theta_0) \overrightarrow{O_5 \sigma_{\psi_1}}
\]

and

\[
\overrightarrow{O_5 \gamma_0 (\psi_0)} = \cos (\psi_0) \overrightarrow{O_5 \sigma_{\psi_1}} + \sin (\psi_0) \overrightarrow{O_5 \sigma_{\psi_1}}
\]
Using Theorem 1 and Formula (3.7), the spheres represented by \( \gamma_\theta (\theta_0) \) and \( \gamma_\psi (\psi_0) \) are tangent if and only if

\[
\mathcal{L}_{4,1} \left( \overrightarrow{O_5 \gamma_\theta (\theta_0)}, \overrightarrow{O_5 \gamma_\psi (\psi_0)} \right) = 1 \quad (5.10)
\]

Then, we can simplify \( a_\theta \) into

\[
a_\theta = \cos (\theta_0) \mathcal{L}_{4,1} \left( \overrightarrow{O_5 \sigma_\theta}, \overrightarrow{O_5 \sigma_\psi} \right) + \sin (\theta_0) \mathcal{L}_{4,1} \left( \overrightarrow{O_5 \sigma_\psi}, \overrightarrow{O_5 \sigma_\theta} \right)
\]

and \( b_\theta \) into

\[
b_\theta = \cos (\theta_0) \mathcal{L}_{4,1} \left( \overrightarrow{O_5 \sigma_\theta}, \overrightarrow{O_5 \sigma_\psi} \right) + \sin (\theta_0) \mathcal{L}_{4,1} \left( \overrightarrow{O_5 \sigma_\psi}, \overrightarrow{O_5 \sigma_\theta} \right)
\]

Thus, Equation (5.10) is equivalent to

\[
a_\theta \cos (\psi_0) + b_\theta \sin (\psi_0) = 1 \quad (5.11)
\]

which admits solution is and only if \( a^2_\theta + b^2_\theta \geq 1 \). Using Lemma 1, we have

\[
a^2_\theta + b^2_\theta = 1
\]

which is equivalent to

\[
\cos (2\theta_0 + \theta_s) = 2 \frac{1 - B^2}{A^2 + B^2} - \cos (\theta_s)
\]

where \( \theta_s \) is the solution of the system

\[
\begin{cases}
\cos (\theta_s) = \frac{A^2 - B^2}{A^2 + B^2} \\
\sin (\theta_s) = 2 \frac{A B}{A^2 + B^2}
\end{cases}
\]

The relation \( A^2 + B^2 = 1 \) implies

\[
\cos (2\theta_0 + \theta_s) = 2 - 2B^2 - (A^2 - B^2) = 2 - (A^2 + B^2) = 1
\]

and we obtain two solutions \( \theta_0 = -\frac{\theta_s}{2} \) and \( \theta_0 = -\frac{\theta_s}{2} + \pi \).

The relation \( a^2_\theta + b^2_\theta = 1 \) implies the Formula (5.11) is equivalent to

\[
\cos (\psi_s) \cos (\psi_0) + \sin (\psi_s) \sin (\psi_0) = 1 \quad (5.12)
\]
where $\psi_s$ is the solution of the system
\[
\begin{align*}
\cos(\psi_s) &= a_{\theta_0} \\
\sin(\psi_s) &= b_{\theta_0}
\end{align*}
\]
and, finally, from Formula (5.12), we obtain $\psi_s = \psi_0$. ■